Einstein’s Equation

\[ G_{\mu\nu} = 8\pi G (T_{\mu\nu} + \rho_{\Lambda} g_{\mu\nu}) \]
Curvature

General Relativity:

Gravity = curvature of spacetime
Curvature

General Relativity: **Gravity = curvature of spacetime**

How to measure curvature? Tough math in 4 dimensions, start with 2 dimensions

Need definition of curvature which is:

1. **Internal to space**
   - Can’t leave the Universe and look at it from the outside

2. **Local**
   - Gravity acts locally, can’t travel across entire Universe to determine gravity at Earth
Curvature

1. Internal to space
2. Local

**Gaussian Curvature (2-D)**

*Measure curvature by departures from Euclidean Geometry near a single point*

**Examples:**

a) Sum of angles in triangle $\neq 180$ degrees
b) Circumference of circle $\neq 2\pi r$
c) Area of circle $\neq \pi r^2$
d) Etc.
Gaussian Curvature

Example:
A cylinder is NOT curved!!

- cut along axis, flatten = plane, Euclidean geometry applies
Gaussian Curvature

Example:
Moebius Strip is NOT curved!!

cut side to side, untwist, flatten = plane, Euclidean geometry applies
Gaussian Curvature

Example:
A sphere is curved!!

a) Sum of angles in triangle $> 180^\circ$

b) Circumference of circle $< 2\pi r$

c) Sphere cannot be flattened into plane

Problem with maps of Earth
Proof: Gravity = Curved Spacetime

Consider photon moving radially outward from a fixed emitter to a fixed observed in the Solar System

In spacetime
Proof: Gravity = Curved Spacetime

Worldlines 1 & 2 have $x = \text{const}$, both $\perp x$ axis
$\Rightarrow 1 \parallel 2$

Wavecrests move at $c$, so $\Delta x = c \, t$, $45^\circ$ angle
$\Rightarrow$ wavecrests all parallel
Proof: Gravity = Curved Spacetime

Side 1 = c \( P_{em} = \frac{c}{\nu_{em}} = \lambda_{em} \)

Side 2 = c \( P_{obs} = \frac{c}{\nu_{obs}} = \lambda_{obs} \)

Gravitational redshift → Side 2 > Side 1

Opposite sides of parallelogram not equal!!
Proof: Gravity = Curved Spacetime

Opposite sides of parallelogram not equal!!

True on surface of sphere, opposite sides of rectangle are unequal
Gaussian Curvature

1. Internal to space
2. Local

Gaussian Curvature (2-D)

Measure curvature by departures from Euclidean Geometry near a single point

Examples:

a) Sum of angles in triangle $\neq 180$ degrees

b) Circumference of circle $\neq 2\pi r$

c) Area of circle $\neq \pi r^2$

d) Etc.
Gaussian Curvature

Use circumference $C$ of circle

Circle $\equiv$ set of points a fixed distance $r$ from a point $P$

Note: $r$ and $C$ must be measured within surface

Curvature at $P$ is then

$$K \equiv \frac{3}{\pi} \left( \frac{2\pi r - C}{r^3} \right) \biggr|_{r \to 0}$$
Gaussian Curvature

\[ K \equiv \frac{3}{\pi} \left( \frac{2\pi r - C}{r^3} \right) \bigg|_{r \to 0} \]

Sphere:
\[ C = 2\pi r' < 2\pi r \]
\[ (r' = R \sin \theta, \ r = R \theta) \]
\[ K > 0 \]

Problem set: \[ K = +1/R^2 \]

General: \[ |K| = 1/(\text{radius of curvature})^2 \]
Gaussian Curvature

Plane $K = 0$  Sphere $K > 0$  Saddle-shape $K < 0$

(hyperbolic)
Riemann Curvature

Spacetime = 4 dimensional
Curvature given by 4D tensor (Riemann Tensor)
General Relativity: related to 4D Einstein Tensor
Symmetric ($xy = yx$):
   10 numbers
Diagonal:
   Conservation of energy (tt)
   Conservation of momentum (xx, etc.)
6 equations for curvature
Like Gaussian curvature for each pair of coordinates, $K_{xt}$, $K_{yt}$, $K_{zt}$, $K_{yx}$, $K_{zx}$, $K_{zy}$
Gravity = Curvature of Spacetime

Explains:

1. Principle of Equivalence
   Both gravity & acceleration → curved paths in spacetime

2. Motion independent of mass
   Geometry the same for all particles, independent of mass

3. Mach’s Principle (maybe)
   Rest of Universe → gravity → local geometry → inertial frames

4. Non-local (tidal) gravity
   Cannot tell curvature of surface from a single point
Gravity = Curvature of Spacetime
Proper Time and Distance

How to quantify “geometry” and curvature?
Geometry: numbers enter through distances
(e.g., radius & circumference of circle)
Want “covariant” definition → all observers agree
Have someone move along path with a watch, and call out the time

$$τ \equiv \text{“proper time”} \equiv \text{time as measured by observer on path}$$

$$s \equiv ct \equiv \text{“proper distance”}$$
Proper Time and Distance

Includes causality

$s^2, \tau^2 > 0$, “timelike” curve, possible future or past

$s^2, \tau^2 = 0$, photon

$s^2, \tau^2 < 0$, “spacelike”, no causal connection
Proper Time and Distance

How to calculate proper time or distance?
Take a small piece of path (dx, dy, dz, dt)
Length is ds
\[(ds)^2 = \text{quadratic function of } dx, dy, dz, dt \equiv \text{“metric”}\]

Metric of flat spacetime:
\[(ds)^2 = (c \, dt)^2 - (dx)^2 - (dy)^2 - (dz)^2\]
Contains all of special relativity!!
Equation of Motion?

Newton:
1. No force, straight line at constant speed
2. $\vec{a} = \frac{\vec{F}}{m}$

Einstein:
Objects move in “straight lines”
What is “straight line” in curved spacetime?

Euclid: straight line = shortest distance between two points

Objects follow path with shortest proper time $\tau \equiv$ “geodesics”
Alternative Theories to GR

Differ in how much of Principle of Equivalence and/or Mach’s Principle is included
Classical Test of GR

1. Advance of perihelion of Mercury
   In GR, orbits not exactly ellipses
   Only 43”/century due to GR

2. Gravitational redshift
   Lines from white dwarfs (1920s)
   Direct measurement on Earth (Pound-Rebka 1959)

3. Bending of starlight by Sun
   Eddington 1919
   But, all “weak field” tests
   \[ \frac{GM}{r} \ll c^2 \]
Strong Field Tests?

All fun involves strong field gravity
  Black holes
  Cosmology
Need strong field tests! (preview – binary pulsars)
Black Holes
ASTR 2110
Sarazin

Calculation of Curved Spacetime
near Merging Black Holes
Black Holes

Gravity crushes star to a point = **singularity**
Surrounded by "surface" from which nothing can escape = **event horizon**
Schwarzschild Metric for Black Hole

Karl Schwarzschild (1916):
Spherical coordinates. Non-rotating black hole

\[
(ds)^2 = \left( c \sqrt{1 - \frac{2GM}{c^2 r}} \ dt \right)^2 - \left( \frac{dr}{\sqrt{1 - \frac{2GM}{c^2 r}}} \right)^2 - (r \ d\theta)^2 - (r \sin \theta \ d\phi)^2
\]

\[
(ds)^2 = \left( c \sqrt{1 - \frac{R_s}{r}} \ dt \right)^2 - \left( \frac{dr}{\sqrt{1 - \frac{R_s}{r}}} \right)^2 - (r \ d\theta)^2 - (r \sin \theta \ d\phi)^2
\]

\[R_s \equiv \frac{2GM}{c^2} = 3 \text{ km } \left( \frac{M}{M_\odot} \right)\]
Schwarzschild Metric

2 odd places

\( r \to 0, \quad \frac{2GM}{c^2r} \to \infty \)

\( r = R_s, \quad (dt)^2 \text{ term } \to 0, \quad (dr)^2 \text{ term } \to \infty \)

\[
(ds)^2 = \left( c \sqrt{1 - \frac{2GM}{c^2r}} \ dt \right)^2 - \left( \sqrt{1 - \frac{2GM}{c^2r}} \frac{dr}{r} \right)^2 - (r \ d\theta)^2 - (r \sin \theta \ d\phi)^2
\]

singularity

event horizon
Schwarzschild Metric

event horizon ≠ physical singularity, just coordinate singularity
Like North Pole on Earth
Black Hole Physics

Singularity = Doom

Tidal forces $\rightarrow \infty$

Worry:

$\infty$ Physics goes to pot!?
Black Hole Physics

Nothing Escapes Event Horizon

\[(ds)^2 = \left(c \sqrt{1 - \frac{R_s}{r}} \, dt\right)^2 - \left(\frac{dr}{\sqrt{1 - \frac{R_s}{r}}}\right)^2 - (r \, d\theta)^2 - (r \sin \theta \, d\phi)^2\]

\[r < R_s \Rightarrow \frac{R_s}{r} > 1 \Rightarrow \sqrt{1 - \frac{R_s}{r}} \text{ imaginary, } \left(\sqrt{1 - \frac{R_s}{r}}\right)^2 < 0\]

\[(dt)^2 \text{ term negative (normally positive)}\]

\[(dr)^2 \text{ term positive (normally negative)}\]

\[r \leftrightarrow t \text{ radius and time reverse roles!!}\]
Black Hole Physics

Nothing Escapes Event Horizon

"Now, here, you see, it takes all the running you can do, to keep in the same place. If you want to get somewhere else, you must run at least twice as fast as that!"

-- The Red Queen
Cosmic Censorship?

There are no “naked singularities”, uncovered by a horizon.
Black Hole Physics

Nothing Really Happens @ Event Horizon

Infalling person:
  falls through horizon, nothing happens

Outside observer:
  first speed up, then slow down, redshift, fade, hang forever just above
Black Hole Physics

Nothing Really Happens @ Event Horizon

Infalling person:
falls through horizon, nothing happens
Black Hole Physics

Nothing Really Happens @ Event Horizon

Infalling person:
falls through horizon, nothing happens

Outside observer:
first speed up, then slow down, redshift, fade, hang forever just above
Infall into Black Hole

infall slows, $v \to 0$, $\infty$ redshift, fades away, hovers forever just above horizon
Black Hole Physics

Orbits near Black Hole

In general, complex. Consider circular orbits

Newton: any radius possible

GR: No stable circular orbits $r \leq 3 R_S$
Black Holes Have No Hair

No matter what they swallow, they only retain:
Mass M, Charge Q, Spin Angular Momentum J

Interstellar matter, very high electrical conductivity
➔ “shorts out” charge

Astrophysical Black Holes:
Mass M, Spin Angular Momentum
Spinning Black Holes

Kerr Metric (Boyer-Linquist Coordinates)

\[
(d \text{s})^2 = \left(1 - \frac{R_s r}{\rho^2}\right)(c \: dt)^2 - \frac{\rho^2}{\Delta}dr^2 - (\rho \: d\theta)^2 \\
- \left(r^2 + \alpha^2 + \frac{R_s r \alpha^2}{\rho^2} \sin^2 \theta\right)(\sin \theta \: d\phi)^2 + \frac{2R_s r \alpha \sin^2 \theta}{\rho^2}c \: dt \: d\phi
\]

\[R_s = \frac{2GM}{c^2}\] Schwarzschild radius

\[\alpha \equiv \frac{J}{Mc}\] (units of length)

\[\rho^2 \equiv r^2 + \alpha^2 \cos^2 \theta\] (units of length)

\[\Delta \equiv r^2 - R_s r + \alpha^2\] (units of length^2)
Spinning Black Holes

Kerr Metric

Exterior Solution

Event Horizon

Oblate spheroid "ergosphere"

Can escape, but must rotate with black hole

top view

side view
Spinning Black Holes

Kerr Metric

Interior Solution
- Ring singularity
- Inner Horizon!?
- But interior solution is unstable
Spinning Black Holes

Kerr Metric

Worm holes??

Unstable $\Rightarrow$ closes as soon as anything enters it!
Energy from Black Holes

Penrose Mechanism

Rotating BH
  Throw something out backwards

\[ E_{\text{out}} \approx E_{\text{in}} + \Delta mc^2 \] !!

May not be important in nature (too complicated), but …
Solution to Mankind’s Problems

Biggest problems:

- Get rid of garbage, pollution
- Make energy
- Build city around rotating BH
Energy from Black Holes

Natural Method

- Drop matter into BH carelessly
- Bump into, rub against other matter
- Friction $\rightarrow$ KE $\rightarrow$ heat $\rightarrow$ light
- Virial Thm.

Heat $\approx (1/2) |PE| \approx (1/2) \frac{GMm}{R_s}$

$\approx (1/2) \frac{GMm}{(2GM/c^2)}$

$E_{\text{out}} = \text{Heat} \approx mc^2/4$ !!!

Accretion by BHs is biggest important energy source in Universe
Black Hole Physics

Black Hole Surface Areas

When BHs merger, mass can increase or decrease

Total surface area of BH horizons always increases in GR

2\textsuperscript{nd} Law of BH Dynamics

Entropy always increases

\[ S = \frac{c^3 k}{4\hbar G} A \quad \text{Entropy} \]

\[ T = \frac{\hbar c^3}{8\pi kG M} \quad \text{Temperature} \]