Selection of Homework Questions

Topic 16: Cosmological Context

(1) Energy in the CMB

a. A spherical human (!) of radius 0.5m floats in the IGM, exposed only to the CMB. Their density and heat capacity are those of water (1 gm cm^{-3}; 4.2 J gm^{-1} K^{-1}). Starting at normal body temperature (36^{o}C), if they absorb all radiation and emit none, how long is it before they boil:
   (a) at the current epoch
   (b) at z = 1100, the time of the creation of the CMB.
   (make sure you get the right factors of π for the absorbed energy!)

b. For a thermal gas of number density n, and mean particle velocity v, how many particles strike unit area per second? Using the current baryon content Ω_b = 0.04 and Hubble Constant H_o = 72 km s^{-1} Mpc^{-1}, compare the thermal energy impinging on our unlucky subject from baryons and photons at the time of the CMB, when $T_{gas} = T_{photons}$ (assume the gas is fully ionized hydrogen). Which contributes more to the heating, electrons or protons, and why?

(2) Geometry in Curved Spaces :

a. **Derive** the metric for a spherical 2-D surface of radius R using the coordinates r, θ (r is **along** the surface, of course)

   (a) What is ds along a "line of latitude" at constant r for a small sector dθ of a circle. Hence, what is the circumference of a circle radius r.

   (b) If you can measure lengths to 1 km accuracy (eg a car odometer over a long journey), how big must r be to detect the curvature of the earth ($R_\oplus \approx$ 6000 km)?.

   (c) What's the relationship between the area of a spherical triangle and the sum of its interior angles? If you can measure angles to 1 arcmin, how big (side length) must an equilateral triangle be to detect the curvature of the Earth?

b. 3-D spatial curvature within a region of mass density ρ is roughly $R_c^2 \sim 3\, c^2 / (8\pi G\rho)$. Show that this can be re-expressed as $R_c \approx c \times t_{dyn}$, where $t_{dyn} \sim r_{orb} / v_{circ}$ is the dynamical time for a circular orbit about the system.

   (a) Hence estimate the **local** spatial radii of curvature (i) near the Earth's surface, (ii) within the solar system (eg near the Earth's orbit), (iii) within the galaxy (eg near the sun's orbit), (iv) within the universe.

   (b) Does this curvature affect metrology with the following levels of accuracy: (a) two satellite

GPS triangulation to 1 cm on earth; (b) wide angle planetary separations to 1 arcsec; (c) wide angle globular cluster separations to 1 arcsec; (d) angular separations within the local supercluster (50 Mpc) to within 1 arcmin.

(Hint: use the triangle area relation from part c above).

(3) Equations of State

a. Write down the equation for the total (relativistic) energy, E, of a particle of rest mass m_o and momentum p. The de Broglie wavelength of this particle is given by $\lambda = h / p$, which increases with the scale factor, just like light: $\lambda \propto a$. Derive an expression for the equation of state parameter, w, for a gas of these particles, assuming they all have the same m_o and p, and that the total energy density is given by U = nE for n such particles per unit volume. Show that in the relativistic limit $w \rightarrow 1/3$ and in the non-relativistic limit $w \rightarrow 0$. (Recall: pressure $P = w\rho c^2$ where ρ is the total energy density). [Ryden: Q 4.5].

b. Imagine if, for some bizarre reason, all the matter in the Universe (both baryonic and dark matter) suddenly annihilated to become photons. A stupendously spectacular, but energy conserving, conversion. How would this change the future evolution of the scale factor, both in the short term, and the long term?

c. Compare the histories of the scale factor for two slight variants on our actual universe:

   (a) Perfect matter-antimatter symmetry led to full annihilation in the first second.
   (b) Something suppressed annihilation, so all CMB photons are instead proton/electron pairs.

(4) Observing De/Acceleration

a. Can we measure de/acceleration **directly** by watching for a gradual change in z over time for an object? Start with the fundamental definition of redshift: $1 + z = a(t_o)/a(t_e)$. Now, as time passes, **both** scale factors change, since obviously $a(t_o)$ is increasing, but so is $a(t_e)$ since the light we see sets out a little later. Now find dz/dt by differentiating (1 + z) w.r.t. time. What are $da(t_o)/dt$ and $da(t_e)/dt$? Substitute in for these, using E(z), to find an expression for dz/dt.

b. Currently, the highest accuracy with which spectral features can be measured is ~0.01 A at 8000A. For an Einstein de-Sitter universe, how long must we wait between observations to detect a change in redshift of a spectral feature at a redshift of (a) 1, (b) 4, (c) 8. Assume that in each case, a spectral feature is identified near 8000A. Compare your answer for z = 1 with the concordance model.

c. Of course, the measured z also has a peculiar velocity component, z_{pec}, which has built up over a Hubble time. Typically, will these peculiar accelerations thwart our ability to observe the cosmic de/acceleration?

(5) Proper Distances

A galaxy is at redshift z = 1. What are the three proper distances: $d_p(t_o)$, $d_p(t_e)$, $c(t_o - t_e)$, to the galaxy in a single component flat universe filled with (a) matter, (b) radiation, (c) vacuum?

(6) Concordance Model

1. Use the concordance model parameters ($\Omega_m = 0.27$, $\Omega_v = 0.73$, $\Omega_{rel} = 8.4 \times 10^{-5}$, $H_o = 72$), to plot the following as a function of redshift. Use three separates graphs for a, b, c. Plot linear z ranges of 0 - 5 for a and b and log z from -1.0 to 5.0 for c. Mark on each plot the times of matter/vacuum equality (and for c, the time of relativistic/matter equality).

   a. $d_p(t_o)$; $d_p(t_e)$; $c(t_o - t_e)$

   b. Angular diameter distance, D_A;
   Luminosity distance, D_L.

   c. Hubble parameter $H(z)$;
   The velocity history, $v(z)$, of a galaxy which is **currently** at 1 Mpc [ie $a \times H(z)$].

   You will need to write a routine to evaluate E(z) and its integral. I suggest you make use of the integrator qromb (which also calls trapzd and polint) in Numerical Recipes.

2. You observe a magnitude 28.0 supernova at z = 3, located 1.2 arcsec from its host galaxy nucleus. Spectra show emission line widths implying an expansion speed of 10^4 km/s. What's the absolute magnitude of the supernova; its projected distance from the galaxy nucleus; and what angular velocity (in mas/yr) does the expansion velocity correspond to? (Ignore K corrections in your calculation of the absolute magnitude).

(7) The Age Problem

The observed age of the oldest globular clusters, 13 Gyr, and the Hubble Constant, 72 km/s/Mpc together place an interesting constraint on the density of a pure matter Universe.

a. What is the Hubble time: $t_{H,o}$?

b. What is t_{age} for $\Omega_m = 0$ (empty) and 1 (flat; Einstein-de Sitter).

c. Write an integral equation for t_{age} and solve it (numerically) to find $\Omega_{m,o}$ such that t_{age} = 13.0 Gyr -- the age constraint from GCs. Alternatively, you may solve the parametric equations for an open pure matter universe, finding η_1 such that $a(\eta_1) = 1$, and then finding $t_{age} = t(\eta_1)$.

d. Measurements of cluster dynamics suggest $\Omega_{m,o} \approx 0.3$. What age does this give?

Do you now see why there was an "age problem" with pure matter models.

(8) Vacuum Energy's Accelerating Expansion

The fact that a vacuum dominated universe accelerates in its expansion is often viewed as deeply puzzling. Phrases like "repulsive gravity", or "negative pressure" only confuse one's intuition. This question aims to provide an intuitive understanding. Although the analysis isn't rigorous, since it is essentially Newtonian, it nevertheless captures the essence of what's going on.

a. Consider a sphere of (non-relativistic) matter: radius R, mass M, and uniform density ρ_m. What energy resides in the gravitational field of this sphere, i.e., what is it's gravitational binding energy, U_{grav}? Why is the sign of U_{grav} negative?
b. Consider an expansion of this sphere by ΔR. Recalling that matter is conserved (i.e. M is constant), what is dU_{grav}/dR? What is its sign?
c. When things naturally "fall", the motion is always such that U_{grav} becomes MORE NEGATIVE. Since energy is conserved, POSITIVE energy is released, usually kinetic energy (objects accelerate "downwards"). With this in mind, answer the (obvious) question: when you release the sphere of matter, in which direction does it fall, inwards (to smaller R) or outwards (larger R).
d. Now replace the matter in our sphere by vacuum, with uniform density ρ_v. Recall that the strange thing about vacuum energy/density is that it is CONSTANT -- when you expand a sphere of it, more of it appears in the new shell. Repeat the evaluation of dU_{grav}/dR but this time subject to the condition that ρ_v is constant, not M. What is the sign of dU_{grav}/dR?
e. For a first guess, then, if we let go of this new sphere of vacuum, which way will it "fall" -- inwards (to smaller R) or outwards (to larger R)?
f. Not so fast, we've forgotten something important. It requires energy to "make" the new shell of vacuum: $4 \pi R^2 dR \rho_v c^2$. Find the expression for dU_{tot}/dR where $U_{tot} = U_{grav} + Mc^2$.
g. Notice that the condition for "infall" or "outfall" depends on the size of the region, R. What is this condition, expressed first in terms of the gravitational radius of the sphere, R_g (defined as $R_g = GM/c^2$), and then in terms of the vacuum density, ρ_v. For a vacuum density equal to that of water, how big must the sphere be before it continues to expand, making more and more "water" as it does so?
h. Assume our sphere represents an expanding cosmological volume with critical density $\rho_v = 3H^2/8\pi G$. Show that the critical radius for "runaway expansion" is roughly r_H, the Hubble radius.
i. For a related calculation, find the radius of a critical density sphere such that its negative gravitational energy is equal (in magnitude) to its positive rest-mass-energy. What is the TOTAL energy of such a sphere?
j. Although we've applied this discussion to present day dark energy, transfer these ideas to the epoch of inflation, and describe how at very early times, a small region of dense vacuum might launch the "big bang", creating a huge massive expanding universe from essentially nothing.
