AST 554 - Final Exam: Part 1

Due – May 4, 2009

Note: These three problems serve as the first three problems of the AST 554 final exam. Please finish them and turn them in the day of the exam (May 4, 2009). The problems are not difficult - they are structured to walk you through basic derivations. Note also that an "Introduction to Cosmology" set of slides have been posted, but aside from providing background information and putting the three problems in context, the slides are not needed to solve these problems.

Question 1

The first and third Friedmann equations have the form

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8}{3}\pi G\rho - \frac{kc^2}{R^2} \tag{1}$$

and

$$\dot{\rho}c^2 = -3\frac{\dot{R}}{R}(\rho c^2 + P), \tag{2}$$

where R is the scale factor of the Universe, $\dot{R}$ is the time derivative of the scale factor, k is the curvature, ρ is the density, and $\dot{\rho}$ is the time derivative of the density.

We want to derive $R(t)$ in the radiation and matter-dominated eras of the Universe. The radiation-dominated era occurred early in the history of the Universe when ultra-relativistic particles were the dominant source of pressure.

(a)(5 points) In the matter dominated case, the pressure from all sources is much less than the density term, i.e., $P << \rho c^2$. Given this, use equation (2) to solve for the density as a function of scale factor.

(b)(5 points) Consider now a time in the matter-dominated era when R was sufficiently small such that the curvature term in equation (1) is much less that the energy density term, i.e., $8\pi G\rho/3 >> |kc^2/R^2|$. Given this, use the $\rho(R)$ expression from 1(a) and equation (1) to show that $R \propto t^{2/3}$.

(c)(5 points) In the radiation-dominated era, the dominant energy density comes from ultra-relativistic particles (e.g., photons), which obey $P = \rho c^2/3$. As in 1(a) and 1(b), solve for the ρ as a function of R, then show that $R \propto t^{1/2}$.

Question 2

Consider again Equation 1. We are concerned with examining certain consequences for having a positive, zero, and negative curvature Universe, i.e., $k = +1, 0$, and -1, respectively.

(a)(5 points) In the case where $k = 0$, the Universe will continue to expand, only coming to a halt at $time = \infty$. Use equation (1) to solve for the density, ρ. This density is the known as the critical density Universe.

(b)(5 points) In the case where $k = +1$, the Universe will eventually halt the expansion, and begin to contract. Use equation (1) to solve for the scale factor R, as which the expansion halts.

(c)(5 points) In the case where $k = -1$, the Universe will continue to expand forever. After sufficient time, the curvature term of equation (1) will dominate over the density term. Solve for $\dot{R}$ when $|kc^2/R^2| >> 8\pi G\rho/3$. This gives the expansion rate of the Universe at late times.

Question 3

In this problem, we will estimate the baryon-to-photon number density.

(a)(5 points) First, let us consider the photon number density due to the cosmic microwave background, which can be expressed as,

$$n_{\gamma,\mathrm{CMB}} \sim \frac{aT^4}{2.8kT},$$

where a is the radiation constant, k is the Stefan-Boltzmann constant, and T = 2.7 K in the present epoch. What is $n_{\gamma,\mathrm{CMB}}$ (in units of cm^{-3})?

(b)(5 points) Estimate the present number density of photons from stars, $n_{\gamma,*}$. For this, that the average density of galaxies to be $n_{\mathrm{gal}} \sim 10^{-2}$ Mpc^{-3}, $L_* \sim 10^{10}$ $\mathrm{L}_\odot$, the average photon energy to the $h\nu_{\mathrm{opt}} \sim$ 1eV, and the age of the Universe to be 14 Myr. For this approximation, we're ignoring the expansion of the Universe, curvature of space, evolution in L_* and n_{gal}. How does this approximation compare with the CMB photon number density?

(c)(5 points) The baryon density of the universe is 4% the critical density. Use the solution to 2(a) to derived the baryon number density. Adopt an average baryon mass = the proton mass.

(d)(5 points) Thus, derive the baryon-to-photon number density ratio.